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On the r-domination number of a graph

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Abstract

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For $r > 0$, let the r-domination number of a graph, d_r, be the size of a smallest set of vertices such that every vertex of the graph is within distance r of a vertex in that set. This paper contains proofs that every graph with a spanning tree with at least $n/2$ leaves has $d_r \leqslant n/(2r)$; this compares with the easy upper bound of $\lceil n/(2r+1) \rceil$ for graphs with Hamiltonian paths.

1. Introduction

We consider properties of connected graphs that ensure relatively small r-domination number. In particular we consider the interrelation of minimum degree and this parameter, and pose a conjecture about Eulerian graphs that originally motivated this work.

Definition. For $r > 0$, a subset D_r of vertices of a graph G is called an *r-dominating set* if every vertex of G is within distance r of a vertex in D_r. The *r-domination number* of a graph, d_r, is the size of a smallest r-dominating set.

(The *distance* between two vertices is the minimum number of edges in a path joining them.) The 1-domination number is also known as the domination number [2]; the r-domination number has also been called the r-covering number [8] and an r-dominating set an r-basis [11]. In general we follow the terminology of [2].

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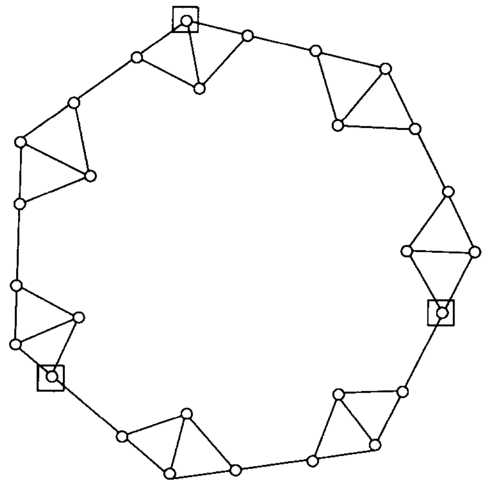


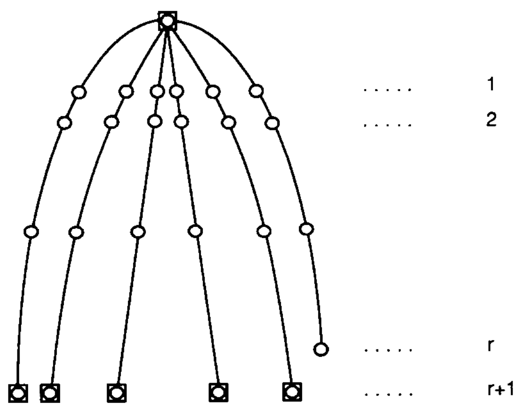
r = 3, n = 28

r-dominating number = 3

Fig. 1.

Figs. 1 and 2 show three examples of graphs, their r-domination numbers, and minimum r-dominating sets (the boxed vertices).

(Notice that Fig. 2(b) shows that an algorithm that is greedy with respect to vertex degrees will not necessarily produce a minimum r-dominating set: Placing the root vertex in D_r results in a larger set of size $2w + 1$.) It is well known that determining the r-domination number of a graph, even for $r = 1$, is an NP-complete problem; however, there is a polynomial-time algorithm for determining the r-domination number of a tree [3, 11].



w-1 paths of r+1 edges each, plus one of r edges

The r-domination number is w = n/(r+1).

Fig. 2(a).

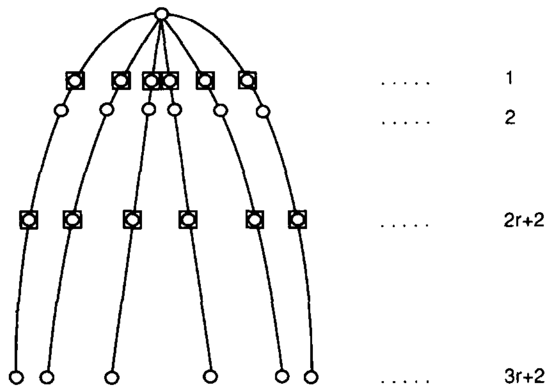


Fig. 2(b).

Ore [10] first proved that every n-vertex graph with minimum degree at least 1 has $d_1 \leq n/2$; this bound has recently been improved to $2n/5$ in [9] for all connected graphs of minimum degree at least 2 with 7 exceptions. More generally, every connected graph with n vertices has $d_r \leq n/(r+1)$ if $n \geq r+1$, and this bound is best possible for graphs similar to those of Fig. 2 [8]. We prove that if a graph contains a spanning tree with no vertex of degree two, known as a HIST, or if it contains a spanning tree with at least $n/2$ leaves, then the graph has r-domination number roughly half as large, $d_r \leq n/(2r)$. Griggs and Wu [5] have recently shown that the latter sort of spanning tree is always present, in a connected graph of minimum degree at least 5. As shown in [1], HISTs are always present in graphs of minimum degree $4\sqrt{2n}$ and in planar triangulations. However, the graphs shown in [5] (respectively, in [1]) that have minimum degree 4 (resp., $\sqrt{n/2}$) and do not contain a spanning tree with $n/2$ leaves (resp., a HIST), also have $d_r \leq n/(2r)$. It may be that minimum degree three is sufficient to ensure the latter bound. There are graphs of minimum degree 2 with $d_r > n/(2r)$; for example, add to each long branch of the graphs in Fig. 2 a vertex adjacent to the last two vertices.

This work originated in [6] where it was shown that an n-vertex triangulation of a surface of genus g has a set of at most $26\sqrt{gn} - 13\sqrt{n/g}$ vertices whose removal leaves a planar graph. The $O(\sqrt{gn})$ bound is known to be best possible, but the constants could be reduced (by about a quarter) if the following conjecture was established.

Conjecture. Every n-vertex Eulerian graph has $d_r \leq \lceil n/(2r) \rceil$.

It may be that that the $n/(2r)$ bound is achieved by 2-connected graphs or 2-edge-connected graphs. Djidjev and Venkatesan [4] have announced significantly better constants in the bounds on the size of a planarizing set, independent of this conjecture, but the conjecture still intrigues.

2. Preliminary results

In this section we give simple proofs that for every connected graph $d_r \leqslant n/(r+1)$, and that a graph containing a Hamiltonian path or, at the opposite extreme, a spanning tree with no vertex of degree two has $d_r \leqslant \lceil n/(2r+1) \rceil$ or $d_r \leqslant (n-2)/(2r)$, respectively. The latter type of spanning tree is known as a *homeomorphically irreducible spanning tree*, or *HIST*.

Definitions. Let T be a tree with root t. Then the vertices of T are partitioned into *distance classes* or *levels* L_i where $L_0 = \{t\}$ and, for $i > 0$, L_i contains precisely those vertices at distance i from t. The *radius* of the rooted tree T is the maximum distance from a vertex to t, or equivalently the largest index i for which L_i is non–empty. The *parent* of a vertex v in L_i, $i > 0$, is the unique vertex w in L_{i-1} to which v is adjacent. We also say that v is a *child* of w. A *descendant* (respectively, an *ancestor*) of w is a vertex v_k for which there is a path of vertices, $w = v_0, v_1, v_2, \ldots, v_k$, such that for $i = 1, 2, \ldots, k$, v_i is a child (resp., the parent) of v_{i-1}. If $v_1, v_2, \ldots, v_k$ form a chain of ancestors of w as in the previous definition, then v_i is said to be the *i-ancestor* of w, for $i = 1, 2, \ldots, k$.

We use a basic greedy approach to obtain some initial results on the r-domination number. The first result of the next proposition has been established using alternative techniques by Meir and Moon [8].

Proposition 1. *For every connected n-vertex graph* $d_r \leqslant \max\{\lfloor n/(r+1) \rfloor, 1\}$. *Every n-vertex graph that contains a HIST has* $d_r \leqslant \max\{\lfloor (n-2)/(2r) \rfloor, 1\}$.

Proof. Let T be an arbitrary spanning tree of G, rooted at a center of T, that is, rooted to minimize its radius, called s. Both results are clearly true if $s \leq r$, for then G has $d_r = 1$. We assume $s > r$ and so $n \geqslant 2r + 2$.

Pick a vertex z at distance s from the root, let x be its r-ancestor, and let T' be the subtree of all descendants of x (in T). Since T', rooted at x, has radius $r < s$, it contains at least $r+1$ vertices. Its removal leaves a connected graph with at most $n - r - 1 \geq r + 1$ vertices, which by induction has $d_r \leqslant (n-r-1)/(r+1)$. The corresponding r-dominating set together with x forms an r-dominating set of G of size at most $n/(r+1)$.

Suppose that T is a HIST of radius $s > r$ and so $n \geqslant 4r + 2$. (T contains a path of at least $2r + 2$ vertices with all but two vertices of degree at least three.) Then the subtree T', as defined in the previous paragraph, has at least $2r + 1$ vertices since it has radius r. (Note that removing T' might create a vertex of degree 2 in the remaining tree.) Removing all vertices of T' except for x from T leaves a HIST with at most $n-2r \geq 2r+2$ vertices, which by induction has $d_r \leq (n-2r-2)/2r$. An r-dominating set of this size together with x forms an r-dominating set of G of size at most $(n-2)/(2r)$. □

As shown in Fig. 2 there are trees that do not have $d_r \leq n/(2r)$. In fact, the tree consisting of a root attached to $w > 2$ paths of p edges each has $d_r \leq n/(2r+1)$ if and only if, setting t to be the remainder when p is divided by $2r+1$, $0 \leq t \leq r$ and $2r \leq tw$. The r-domination bound for HISTs is best possible as seen by a HIST that is a path of $2r+2$ vertices with a vertex of degree 1 attached to each nonleaf vertex. Then $n = 4r+2$ and $d_r = 2$.

Since a graph of minimum degree at least $4\sqrt{2n}$ contains a HIST [1], we have the following.

Corollary 2. *A connected graph of minimum degree at least* $4\sqrt{2n}$ *has* $d_r \leq (n-2)/(2r)$ *if* $n \geq 2r+2$.

Other properties involving diameter and planarity also imply the presence of a HIST.

Proposition 3. *For a graph G with a Hamiltonian path* $d_r \leq \lceil n/(2r+1) \rceil$.

Proof. Let P be a Hamiltonian path of an n-vertex graph G. Form an r-dominating set by selecting the $(r+1)$-st vertex of P and then adding in every $(2r+1)$-st vertex of P, as long as possible. If this set does not r-dominate (e.g., the last vertices of P), add in the final vertex of P. This set has size $\lceil n/(2r+1) \rceil$. □

Rephrased for comparison with Theorem 4, this result says that if G contains a spanning tree with two leaves, then G has $d_r \leq \lceil n/(2r+1) \rceil$; the example of Fig. 2(a) shows that if the spanning tree has three or more leaves, G may not have such a small domination number.

3. Main result

A HIST is a spanning tree with many leaves; in fact, a straightforward count gives that a HIST on n vertices has at least $n/2+1$ leaves. It is reasonable and productive to ask about the r-domination number of graphs with spanning trees with this many leaves. However, to obtain the next result, we look at the complementary set and count the number of *internal vertices*, vertices of degree at least 2 in a spanning tree.

Theorem 4. *If G has a spanning tree T with x internal vertices, then G has* $d_r \leqslant \max\{\lfloor x/r \rfloor, 1\}$.

Proof. First T is divided into rooted subtrees, each of radius at most r, that include all vertices of T and hence G. The roots of these subtrees form an r-dominating set, D_r. Then we count and compare $|D_r|$ with the number of internal vertices of T.

Algorithm. Input a tree T with root t and radius s, and a positive integer r.
If $s \leqslant r$, return $D_r = \{t\}$ and stop.
Otherwise, set $i := 1$
Repeat
- Pick a leaf l_i at maximum distance from t.
  Let x_i be the r-ancestor of l_i, and let T_i be the subtree of T of x_i and all its descendants.
- $D_r := D_r \cup \{x_i\}$.
- $T := T - T_i$.
- $i := i + 1$.

Until all vertices of T are at distance $< r$ from t. If at the end there are vertices of T at distance $< r$, add $x_i = t$ to D_r and set T_i equal to the remaining tree.

(By resetting T to be $T - T_i$, we mean that all vertices and incident edges of T_i are removed from T.) If T is decomposed into j subtrees, for $i = 1, \ldots, j$ we declare x_i to be the root of T_i. All subtrees have radius exactly r except possibly for T_j with radius at most r.

If $s \leqslant r$, then $d_r = 1$, and Theorem 4 is immediate, and so we assume $s > r$ and $j \geqslant 2$. Without loss of generality, we may assume that the root t of T has degree at least 2 (for example, t may be chosen to be a center of T), and it follows that T has $x \geqslant r + 1$ internal vertices.

Each tree T_i of radius r contains at least r internal vertices of T, on the path from l_i to x_i. If T_j also has radius r, then we have, as desired, $jr \leqslant x$.

Otherwise, since T_j contains at least one internal vertex, t,

$$(j-1)r + 1 \leqslant x. \tag{*}$$

Suppose the longest simple path P in T_j contains k vertices. If $k \geqslant r + 2$, then T_j contains at least r internal vertices (all but possibly two endpoints of P). Thus

$$(j-1)r + r \leqslant x, \tag{**}$$

and $j \leqslant x/r$.

If all vertices of T_j are within distance r of x_{j-1} (distance measured within T), then $T_{j-1} \cup T_j$ forms a subtree of radius r, and so removing t from D_r leaves an r-dominating set of at most $j - 1 < x/r$ vertices by (*). Now if $k \leqslant r$, then all vertices of T_j are within distance r of x_{j-1}. Suppose $k = r+1$ and that there is a vertex v at distance $r+1$ from x_{j-1}. Then the path in T_j from v to the neighbor of x_{j-1} contains r internal vertices of the original tree T so that again (**) holds. $\square$

The results of this theorem are best possible: For every r and $x > 0$, there is a tree T' with x internal vertices and $d_r = \lfloor x/r \rfloor$. For example, if $x = kr + s$, $0 \leq s < r$, we pick any tree T with k vertices. We then add to each vertex of T a path of r vertices, and subdivide one edge of T with $s < r$ vertices of degree 2. The resulting T' has $x = kr + s$ internal vertices and $d_r = k$ since the vertices of T form a minimum r-dominating set. We can also show for $x = kr$, $k \geq 2$, that this construction generates all extremal trees except that additional leaves can be attached to the internal vertices.

Since a tree with at least $n/2$ leaves has at most $n/2$ internal vertices, we have the following result.

Corollary 5. *If G has a spanning tree with at least $n/2$ leaves, then $d_r \leq n/(2r)$ if $n \geq 2r$.*

By the results of [5], a connected graph of minimum degree at least 5 has a spanning tree with at least $n/2 + 2$ leaves.

Corollary 6. *A connected graph of minimum degree at least 5 has $d_r \leq n/(2r)$ if $n \geq 2r$.*

The examples that show Griggs and Wu's results to be best possible are similar to that of Fig. 1: Replace each K_4 minus an edge by K_5 minus an edge, forming a 4-regular graph that does not contain a spanning tree with $n/2$ leaves. In general, if G is such a cycle of k copies of K_j minus an edge, $j \geq 4$, then $d_r \leq \lceil n/(2r+1) \rceil$ by Proposition 3. In fact, these graphs have smaller r-domination number, about $3n/(2jr)$.

The following classes of Eulerian graphs are known to have $d_r \leq \lceil n/(2r) \rceil$: a union of cycles with one vertex in common, two vertices joined by $2k$ paths of length p each, a 'path' of cycles $C_1, C_2, \ldots, C_k$ such that $|V(C_i) \cap V(C_{i+1})| = 1$ for $i = 1, 2, \ldots, k-1$, and a similar 'cycle' of cycles. However, the conjecture has not yet been verified for an interesting family of Eulerian graphs except for those of minimum degree at least 6 by Corollary 6; such a family would be those with a vertex of degree two, those of minimum degree four, planar Eulerian graphs, or randomly Eulerian graphs.

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